Adaptive Finite Element Approximation of a Fluid Structure Interaction Problem

Fredrik Bengzon Mats G. Larson Anton Olofsson

Department of Mathematics and Mathematical Statistics, Umeå University

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Motivation, Outline, and Objective

Fluid structure interaction (FSI) is the combination of fluid and solid mechanics into multiphysics problems.

Finite elements are generally used to simulate such problems.

We consider error estimation for finite element approximation of FSI problems.

We will show how to derive error estimates for a *one-way* coupled FSI problem.

The FSI Problem

Consider the flow of water around the keel of a boat.

The water exerts a force on the keel which deforms.

Figure: Boat with a red keel.

Our goal is to predict the deformations on the keel.

Mathematical Model

1. The Stokes equations are solved for the fluid velocity $\mathbf{u}_F$.
2. The stress $\boldsymbol{\sigma}_{SF}$ on the keel surface is computed form $\mathbf{u}_F$.
3. The deformation $\mathbf{u}_S$ of the keel is computed from $\boldsymbol{\sigma}_{SF}$.

We do not consider the effect of displacing the keel on the fluid domain.

Computational Geometry

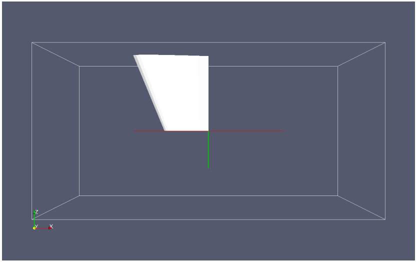

Figure: Side view of keel immersed in channel.

Computational Geometry, cont'd

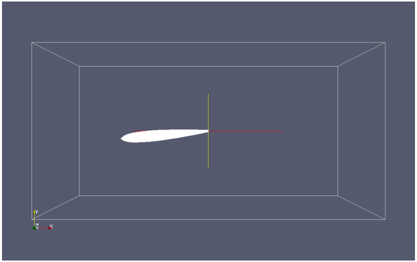

Figure: Top view. Note 5° angle of attack relative centerline.

The Weak Stokes Problem

Find the fluid velocity and pressure
$(\mathbf{u}_F, p_F) \in \mathcal{V}_{F,\mathbf{u}_{F,D}} = \{\mathbf{v} \in [H^1(\Omega_F)]^3 : \mathbf{v}|_{\Gamma_{F,D}} = \mathbf{u}_{F,D}\} \times L^2(\Omega_F)$
such that

$$a_F(\mathbf{u}_F, p, \mathbf{v}, q) = 0, \quad \forall (\mathbf{v}, q) \in \mathcal{V}_{F,\mathbf{0}}$$

where

$$a_F(\mathbf{v}, q, \mathbf{w}, r) = 2\nu(\varepsilon(\mathbf{v}) : \varepsilon(\mathbf{w})) - (q, \nabla \cdot \mathbf{w}) + (\nabla \cdot \mathbf{v}, r)$$

with ν viscosity, and ε the strain rate tensor

$$\varepsilon(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$$

The Weak Elasticity Problem

Find the deformation $\mathbf{u}_S \in \mathcal{V}_{S,\mathbf{0}} = \{\mathbf{v} \in [H^1(\Omega_S)]^3 : \mathbf{v}|_{\Gamma_{S,D}} = \mathbf{0}\}$ such that

$$a_S(\mathbf{u}_S, \mathbf{v}) = l_S(\mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{V}_{S,\mathbf{0}}$$

where

$$a_S(\mathbf{v}, \mathbf{w}) = 2\mu(\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{w})) + \lambda(\nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{w})$$
$$l_S(\mathbf{v}) = (\mathbf{n} \cdot \boldsymbol{\sigma}_F, \mathbf{v})_{\Gamma_{SF}}$$

with μ and λ the Lamé parameters.

The term $(\mathbf{n} \cdot \boldsymbol{\sigma}_F, \mathbf{v})_{\Gamma_{SF}}$ is a traction load representing the fluid pressure on the keel surface Γ_{SF}.

Finite Element Approximation

Let $\mathcal{V}_{F,\mathbf{u}_D,h} = \mathcal{V}_{F,\mathbf{u}_D}$ and $\mathcal{V}_{S,0,h} \subset \mathcal{V}_{S,0}$ be finite element spaces on meshes of the fluid and solid domain with matching nodes on the keel surface Γ_{SF}.

We use Galerkin Least Squares finite element method to solve the Stokes problem: find $(\mathbf{U}_F, P_F) \in \mathcal{V}_{F,\mathbf{u}_D,h}$ such that

$$A_F(\mathbf{U}_F, p_F, \mathbf{v}, q) = 0, \quad \forall (\mathbf{v}, q) \in \mathcal{V}_{F,\mathbf{0},h}$$

where $A_F(\cdot,\cdot)$ is the Galerkin Least Squares version of $a_F(\cdot,\cdot)$

$$A_F(\mathbf{v}, q, \mathbf{w}, r) = a_F(\mathbf{v}, q, \mathbf{w}, r) + (\tau \nabla q, \nabla r)$$

with $\tau > 0$ a stabilization parameter.

Finite Element Approximation, cont'd

The elasticity problem is solved by a standard Galerkin method: find $\mathbf{U}_S \in \mathcal{V}_{S,\mathbf{0},h}$ such that

$$a_S(\mathbf{U}_S, \mathbf{v}) = L_S(\mathbf{v}), \quad \forall \mathbf{v} \in \mathcal{V}_{S,\mathbf{0},h}$$

Here $L_S(\mathbf{v})$ denotes a variational version of $l_S(\mathbf{v})$ defined by

$$L_S(\mathbf{v}) = -a_F(\mathbf{U}_F, P_F, E_F(\mathbf{v}), 0)$$

where E_F is an extension operator from Γ_{SF} onto Ω_F.

In the discrete setting it is better to use $L_S(\mathbf{v})$ than $l_S(\mathbf{v})$ since we avoid explicit differentiation of $\mathbf{U}_F$.

A Posteriori Estimate for the Elastic Problem

To represent the error in a given goal functional $m_S(\cdot)$ on the elasticity problem we introduce the dual problem

$$m_S(\mathbf{v}) = a_S(\mathbf{v}, \phi_S)$$

Setting $\mathbf{v} = \mathbf{u}_S - \mathbf{U}_S$ we get the error representation formula

$$\begin{aligned} m_S(\mathbf{u}_S - \mathbf{U}_S) &= a_S(\mathbf{u}_S - \mathbf{U}_S, \phi_S) \\ &= L_S(\phi_S - \pi\phi_S) - a_S(\mathbf{U}_S, \phi_S - \pi\phi_S) \\ &\quad + l_S(\phi_S) - L_S(\phi_S) \end{aligned}$$

We note that

$$\begin{aligned} l_S(\phi_S) - L_S(\phi_S) &= -a_F(\mathbf{u}_F - \mathbf{U}_F, p_F - P_F, E_F(\phi_S), 0) \\ &= m_F(\mathbf{u}_F - \mathbf{U}_F, p_F - P_F) \end{aligned}$$

This is a modeling error stemming from the use of a computed fluid velocity $\mathbf{U}_F$.

A Posteriori Estimate for Stokes Problem

Obviously, we want to represent the error in the functional $m_F(\cdot)$.

To do so we introduce another dual problem

$$m_F(\mathbf{v}, q) = a_F(\mathbf{v}, q, \phi_F, \chi_F)$$

Setting $\mathbf{v} = \mathbf{u}_F - \mathbf{U}_F$ and $q = p_F - P_F$ we get the error representation formula

$$\begin{aligned} m_F(\mathbf{u}_F - \mathbf{U}_F, p_F - P_F) = &-A_F(\mathbf{U}_F, P_F, \phi_F - \pi\phi_F, \chi_S - \pi\chi_F) \\ &+ (\tau\nabla P_F, \nabla\chi_F) \end{aligned}$$

A Posteriori Estimate for the Coupled FSI Problem

Combining the two error representation formulas we get an error representation formula for the coupled problem

$$
\begin{aligned}
m_S(\mathbf{u}_S - \mathbf{U}_S) = L_S(\phi_S - \pi\phi_S) - a_S(\mathbf{U}_S, \phi_S - \pi\phi_S) \\
- A_F(\mathbf{U}_F, P_F, \phi_F - \pi\phi_F, \chi_S - \pi\chi_F) \\
+ (\tau \nabla P_F, \nabla \chi_F)
\end{aligned}
$$

Each term can be estimated using dual weighted residual estimates and, hence, be kept small with adaptive mesh refinement.

Element Indicators

The next estimate holds on the solid and fluid meshes $\mathcal{K}_S$ and $\mathcal{K}_F$.

$$m_S(\mathbf{u}_S - \mathbf{U}_S) \leq \sum_{K \in \mathcal{K}_S} R_{S,K} W_{S,K} + \sum_{K \in \mathcal{K}_F} \mathbf{R}_{F,K} \cdot \mathbf{W}_{F,K} \equiv \sum_{K \in \mathcal{K}} \eta_K$$

where $\mathbf{R}_{F,K} = [R_{F,K,1},\ R_{F,K,2}]$, $\mathbf{W}_{F,K} = [W_{F,K,1},\ W_{F,K,2}]$ and

$$\begin{aligned}
R_{S,K} &= \|\nabla \cdot \boldsymbol{\sigma}_S(\mathbf{U}_S)\|_K + h_K^{-1/2} \|[\mathbf{n} \cdot \boldsymbol{\sigma}_S(\mathbf{U}_S)]\|_{\partial K \setminus \Gamma_{SF}} \\
&\quad + h_K^{-1/2} \|\mathbf{n} \cdot \boldsymbol{\sigma}_S(\mathbf{U}_S) - \mathbf{n} \cdot \boldsymbol{\sigma}_F(\mathbf{U}_F)\|_{\partial K \cap \Gamma_{SF}} \\
W_{S,K} &= h_K^2 |\phi_S|_{\mathcal{N}(K),2}, \\
R_{F,K,1} &= \|\nabla \cdot \boldsymbol{\sigma}_F(\mathbf{U}_F)\|_K + h_K^{-1/2} \|[\mathbf{n} \cdot \boldsymbol{\sigma}_F(\mathbf{U}_F)]\|_{\partial K \setminus \Gamma_{F,D}} \\
R_{F,K,2} &= \|\nabla \cdot \mathbf{U}_F\|_K \\
W_{F,K,1} &= h_K^2 |\phi_F|_{\mathcal{N}(K),2}, \qquad W_{F,K,2} = h_K |\chi_F|_{\mathcal{N}(K),1}
\end{aligned}$$

with h_K the element mesh size.

Adaptive Algorithm

1. Solve the primal problems, starting with the Stokes equations to obtain the fluid velocity $\mathbf{U}_F$, and then solve the linear elasticity equation with the fluid stress traction load for the displacements $\mathbf{U}_S$.
2. Solve the dual elasticity problem using the user specified goal functional $m_S(\cdot)$ as right hand side for the dual displacement ϕ_S.
3. Use ϕ_S to compute the modeling functional $m_F(\cdot)$, and solve the dual Stokes equations for the dual velocity ϕ_F and pressure χ_F.
4. Compute the error indicators η_K, and use them together with a refinement criterion to select elements that contribute the most to the error. Refine the selected elements.
5. Repeat steps 1-4 until satisfactory results have been obtained.

Fluid Flow and Keel Displacement

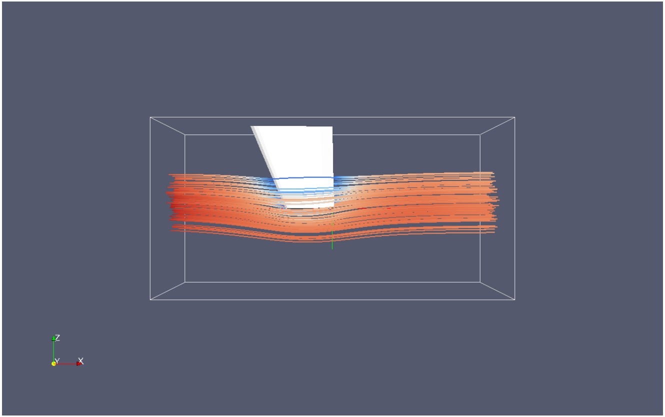

Figure: Side view of streamlines and displacement.

Fluid Flow and Keel Displacement, cont'd

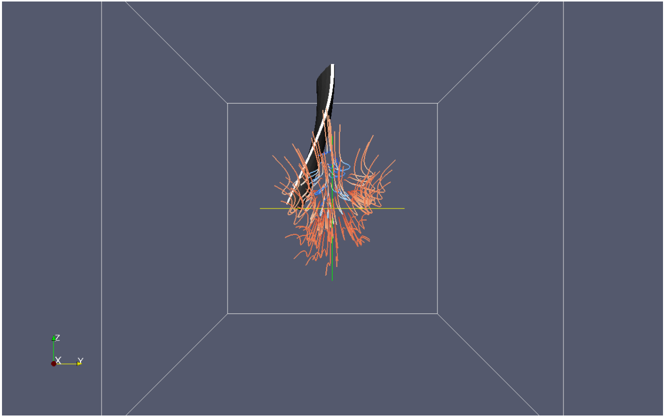

Figure: View from behind the keel facing upstream.

Fluid Flow and Keel Displacement, cont'd

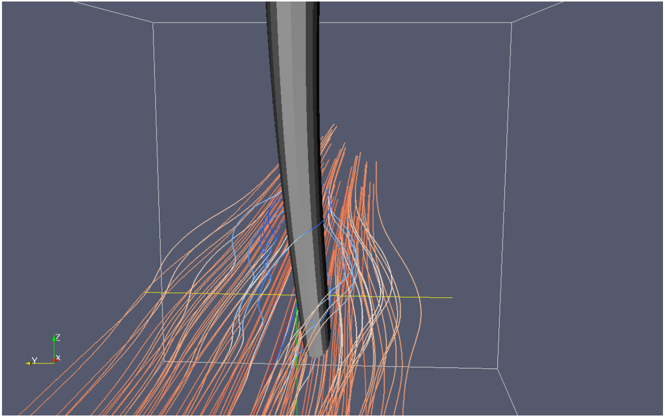

Figure: Zoom of keel.

Example of Goal Functional

Suppose we want control of the displacement u_j, $j = 1$, 2, or 3, near a point $\mathbf{x}_0$ on the keel.

The goal is expressed through the goal functional

$$m_S(\mathbf{v}) = (\mathbf{v}, \psi_j)$$

with ψ_j a narrow Gaussian pulse centered at x_0 and along direction x_j

$$\psi_j = \exp(-c|\mathbf{x} - \mathbf{x}_0|^2)\mathbf{e}_j$$

$\mathbf{e}_j$ is column j of the 3×3 identity matrix.

We choose $\mathbf{x}_0$ as origo, which is on the trailing edge and at the bottom of the keel.

Dual Solution $m_S(\mathbf{v}) = (\mathbf{v}, \psi_2)$

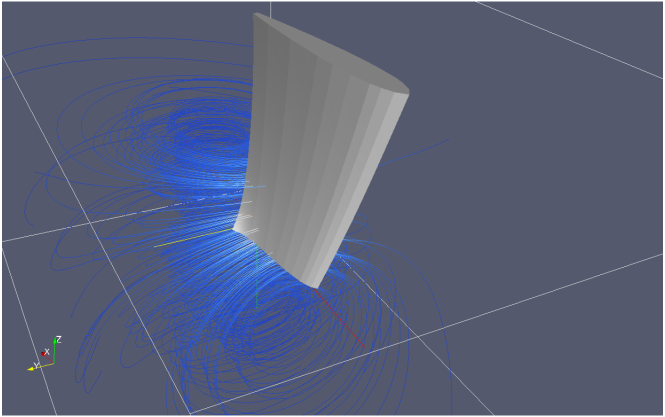

Figure: Side view of dual streamlines and displacement.

Dual Solution $m_S(\mathbf{v}) = (\mathbf{v}, \psi_2)$, cont'd

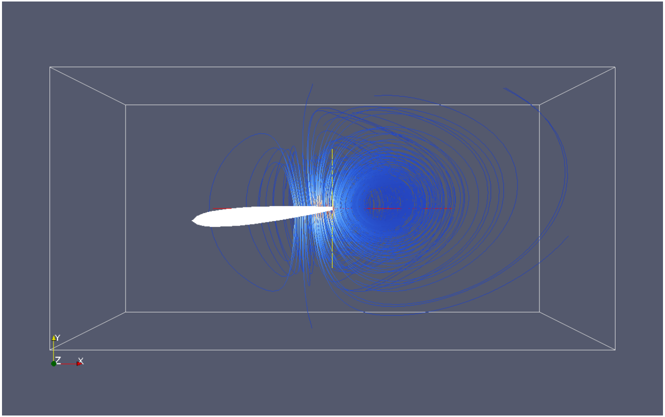

Figure: Top view.

Dual Solution $m_S(\mathbf{v}) = (\mathbf{v}, \psi_2)$, cont'd

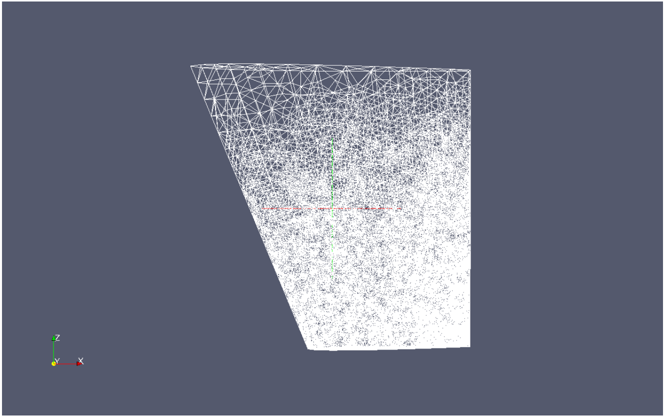

Figure: Mesh density on keel after 10 refinements.

Dual Solution $m_S(\mathbf{v}) = (\mathbf{v},\psi_3)$

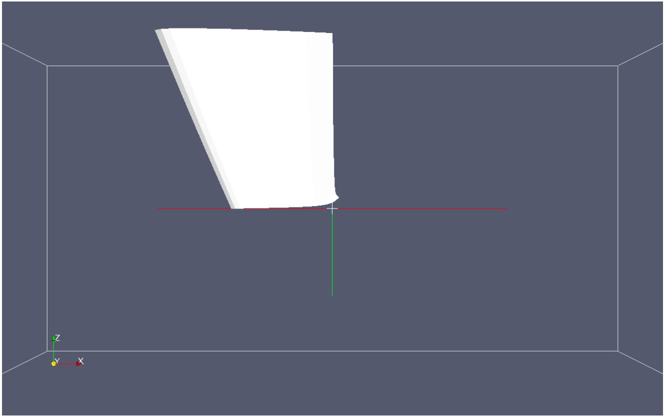

Figure: Dual displacements due to localized load at origo along x_3.

Dual Solution $m_S(\mathbf{v}) = (\mathbf{v}, \psi_3)$, cont'd

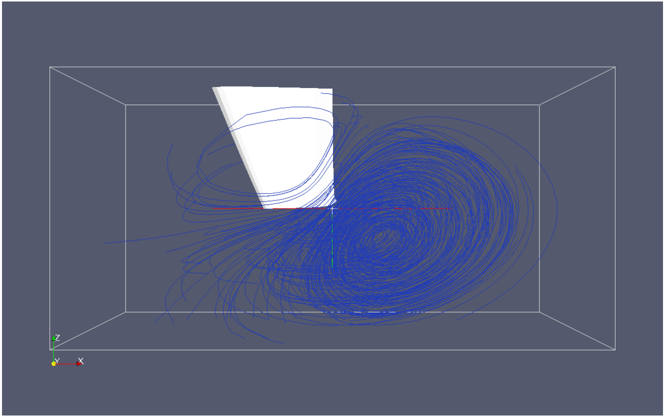

Figure: Side view of dual streamlines and displacement.

Dual Solution $m_S(\mathbf{v}) = (\mathbf{v}, \psi_3)$, cont'd

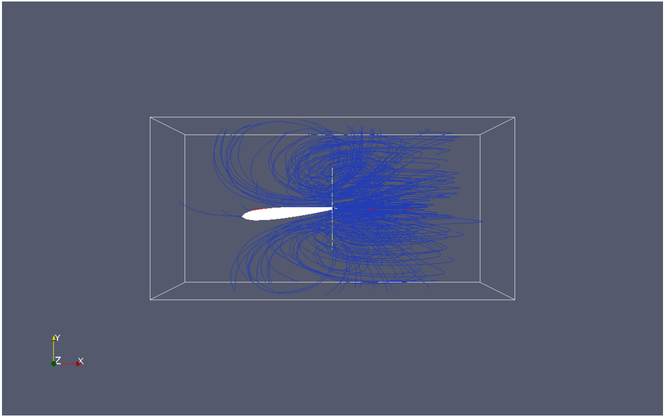

Figure: Top view.

Dual Solution $m_S(\mathbf{v}) = (\mathbf{v}, \psi_3)$, cont'd

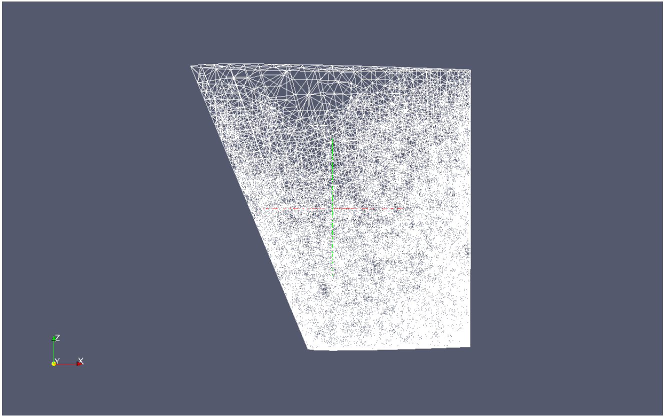

Figure: Mesh density on keel after 10 refinements.